Flow Perpendicular to Mats of Randomly Arranged Cylindrical Fibers (Importance of Cell Models)

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There is considerable interest at the present time in predicting the efficiency of removal of fine particles from fluid streams (aerosol removal, pollution studies) using fibrous filter mats. Several mechanisms influence the removal efficiency (for example, inertial, diffusional, interception, electrostatic, gravitational, and thermal effects). In order to predict the removal efficiency for a given mechanism, an expression for the local hydrodynamic flow field in the vicinity of a typical fiber embedded in the mat is required. The fiber mat is usually represented in the theory by a random arrangement of equal circular cylinders oriented at right angles to the direction of flow although elliptical cylinders have recently been studied (Masliyah, 1973; Masliyah and Duff, 1975).

Three distinct models have been used in removal efficiency studies to estimate the local flow field in the vicinity of a typical fiber, namely, the familiar Brinkman model (Spielman and Goren, 1968) and the widely employed cell models of Happel (1959) and Kuwabara (1959). The solution based upon Brinkman's model possesses a physically inconsistent singularity at $\alpha = 1/2$ (where α denotes the fiber mat concentration). Since the original Brinkman (1947) model, as applied to spheres, possesses an analogous singularity at $\alpha = 2/3$, it is evident that this model is not internally consistent (Neale and Nader, 1974a). The Happel and Kuwabara models are easy to apply and although they generally provide good predictions they are sometimes unfairly criticized because they do not represent any real physical situation (that is, they are not capable of independent existence).

In view of the above considerations, it was considered necessary to compare the predictions of the Brinkman, Happel, and Kuwabara models for flow perpendicular to cylinders with the predictions of the more physically realistic model which was proposed recently to study transport processes within porous media in general (Neale and Nader, 1974a). This new model (Figure 1) possesses a definite conceptual advantage in that it retains all of the principal features of the original fiber mat, namely, a typical reference fiber (of radius R), a region of void space (of outer radius S) associated with the reference fiber, and an exterior region of homogeneous porous medium possessing identical macroscopic properties (porosity and permeability in particular) to the original fiber mat. For macroscopic homogeneity it is necessary that S be defined by $\alpha = R^2/S^2$. Actually, the modeled system is identical with the original system apart from a local rearrangement in the immediate vicinity of the reference fiber (note that the two systems are macroscopically indistinguishable to an outside observer). It should be noted that if no provision is made in the model for a region of void space associated with the reference fiber (that is, if $R = S$), then the model reduces to the simpler Brinkman model discussed earlier. The absence of a region of associated void space creates an unacceptable discontinuity in α near the reference particle, and this is the reason for the singularity in the Brinkman solution (Neale and Nader, 1974a).

The geometric model depicted in Figure 1 has been solved exactly for the case of a swarm of spheres (Neale and Nader, 1974a). The governing equations are the Navier-Stokes equation within the shell of void space and the Brinkman equation (that is, Brinkman's extension of Darcy's equation) within the exterior porous medium. These equations, together with the prevailing set of boundary conditions, are discussed in detail in the original work and elsewhere (Neale and Nader 1973, 1974b). The analytical solution so obtained was rather complicated, requiring an iterative solution, but was in very good agreement with experimental data. It was demonstrated that

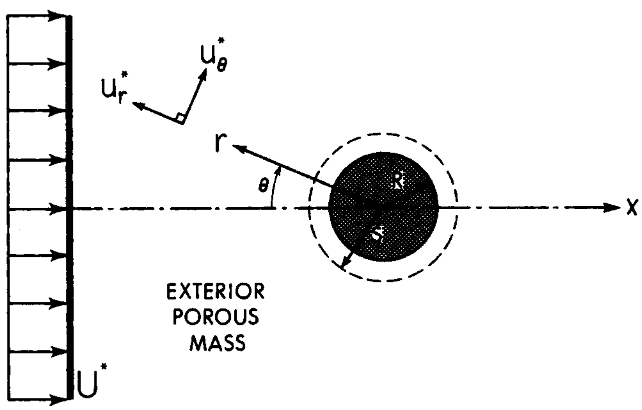

Fig. 1. The proposed model for flow perpendicular to a fiber mat.

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the use of the simple Darcy equation in place of the Brinkman equation in the exterior porous medium region simplified the analytical work considerably, yet yielded a solution which was virtually indistinguishable from that based upon the Brinkman equation (and equally indistinguishable from Happel's solution) for small values of $\alpha (\lesssim 0.2)$. Using the simple Darcy equation an explicit solution, not requiring iteration, could be obtained (a definite advantage). The fact that the solutions based upon the Darcy and Brinkman equations are asymptotic for small α is not merely fortuitous but is to be expected from physical considerations. Thus as α decreases, the thickness of the shell region (S-R) increases, the effects of the shear-induced flow in the boundary layer region just beneath the permeable interface decrease, and the Brinkman equation tends towards the simple Darcy equation, whatever geometry is being considered (see Neale and Nader, 1974b, for further relevant discussion). Because the values of α in commercial fiber mats are usually very small ($\lesssim 0.05$, in order to avoid high pressure drops), it is logical to employ the simple Darcy equation when applying the present model to the case of highly porous fiber mats. (Note that if using this model for $\alpha \gtrsim 0.2$, the Brinkman equation should be employed within the exterior region.)

THEORY

Usually the flow Reynolds number (based upon fiber diameter) is low and creeping flow may be assumed. The governing equations, in terms of stream functions ψ and ψ^*, are therefore

$$\nabla^4 \psi = 0 \quad \text{(Navier-Stokes)} \quad R \leq r \leq S \tag{1}$$

$$\nabla^2 \psi^* = 0 \quad \text{(Darcy)} \quad S \leq r < \infty \tag{2}$$

where

$$\nabla^2 \equiv (\partial^2/\partial r^2) + r^{-1}(\partial/\partial r) + r^{-2}(\partial^2/\partial \theta^2) \tag{3}$$

in cylindrical coordinates $[r, \theta, z]$ with no z-dependence, and $\nabla^4 \psi \equiv \nabla^2(\nabla^2 \psi)$.

The boundary conditions are equivalent to those stipulated when applying the proposed model to spheres, namely,

$$u_r = 0, \quad u_\theta = 0 \quad \text{at} \quad r = R \tag{4}$$

$$u_r = u_r^*, \quad u_\theta = u_\theta^*, \quad p = p^* \quad \text{at} \quad r = S \tag{5}$$

$$u_r^* \to -U^* \cos\theta \quad \text{as} \quad r \to \infty \tag{6}$$

If using the Brinkman equation $[\nabla^4 \psi^* - (1/k)\nabla^2 \psi^* = 0]$ instead of Equation (2), it is necessary to satisfy the additional boundary condition: $\tau = \tau^*$ at $r = S$. This condition is not permissible (in fact it is superfluous) when using the Darcy equation as this equation cannot account for the mechanism of macroscopic shear in a porous medium. The physical implications inherent in Equations (1) to (6) are discussed in detail elsewhere (Neale and Nader, 1973, 1974a, 1974b). Their solution is

$$\psi = RU^*[A\chi^3 + B\chi + C\chi^{-1} + D\chi \ln \chi]\sin\theta \tag{7}$$

$$\psi^* = RU^*[\chi + E\chi^{-1}]\sin\theta \tag{8}$$

where

$$D = \frac{6 - 4\alpha - 2\alpha^2 + 16\xi\alpha}{-2 + 4\alpha - 2\alpha^2 + 8\xi\alpha(1 - \alpha) - (1 - \alpha^2 + 8\xi\alpha) \ln \alpha} \tag{9}$$

and

$$A = \alpha(2 + D \ln \alpha)/(4 - 4\alpha), \quad C = (2A + D)/2,$$

$$B = -A - C, \quad E = (1 + 8A\xi - 2D\xi\alpha)/\alpha,$$

$$\xi = k/R^2, \quad \chi = r/R$$

The drag force per unit length of fiber W is given by (Batchelor, 1967)

$$W = 4\pi\mu U^* D \tag{10}$$

Moreover, the net drag force per unit volume may be equated to the pressure gradient causing flow

$$W/\pi S^2 = -dp/dx = \mu U^*/k \tag{11}$$

the last expression following from Darcy's Law as applied to the overall system. We thus obtain

$$\xi = 1/(4\alpha D) \tag{12}$$

and D may now be identified as a dimensionless pressure gradient or a dimensionless drag force, differing by a factor of 4π from the traditional form of the dimensionless drag force (Kirsch and Fuchs, 1967). Equations (12) and (9) provide the following solution for D (valid for small α):

$$D = \frac{(2 - \alpha - \alpha^2 + \ln \alpha) + \sqrt{(\ln \alpha - \alpha + \alpha^2)^2 - 4(1 - \alpha)^3}}{-2 + 4\alpha - 2\alpha^2 - (1 - \alpha^2) \ln \alpha} \tag{13}$$

Knowing D, the dimensionless permeability $\xi = k/R^2$ is easily determined from Equation (12).

DISCUSSION

The corresponding predictions for D according to the Happel and Kuwabara models are, respectively,

$$D_H = -2/[\ln \alpha + (1 - \alpha^2)/(1 + \alpha^2)] \tag{14}$$

$$D_K = -4/[2 \ln \alpha + 3 - 4\alpha + \alpha^2] \tag{15}$$

The predictions of Equations (13), (14), and (15) are compared in Figure 2 for the range of α values commonly encountered in commercial fiber mats (that is, $\lesssim 0.2$). Also included is the prediction for the dimensionless pressure gradient obtained by Spielman and Goren (1968) using the original Brinkman model, but the solution in this case constitutes an inconvenient implicit relationship

$$D = \frac{2\sqrt{\alpha}\, K_1(2\sqrt{\alpha D})}{(1 - 2\alpha)\, K_0(2\sqrt{\alpha D})} \tag{16}$$

where K_1, K_0 are modified Bessel functions of the second kind. It will be noted that Equation (16) possesses a phys-

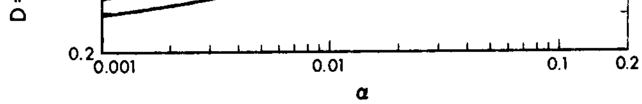

Fig. 2. Theoretical predictions for the dimensionless pressure drop [B = Brinkman model; H = Happel model; K = Kuwabara model; - - - = present model].

ically inconsistent singularity at $\alpha = 0.5$ so this model cannot be regarded as being fundamental in nature. This difficulty is resolved in the present model by the inclusion of a region of void space associated with the reference particle.

It is instructive to note that the predictions of the present model are asymptotic with those of Happel for small α (the differences being 2.5%, 1.4%, and 0.3% at $\alpha = 10^{-3}$, 10^{-4}, and 10^{-8}, respectively). This asymptotic behavior occurs not only in the present case involving cylinders but also in the case of spheres (Neale and Nader, 1974a), indicating further that the two models are fundamentally related.

It should be mentioned that the predictions of Kuwabara's model for the pressure drop across parallel staggered cylinders are in excellent agreement with experimental values (Kirsch and Fuchs, 1967). Moreover, in similar systems of parallel cylindrical fibers the diffusional collection efficiency of aerosol particles is predicted well by Kuwabara's model—such an efficiency is related to $D^{1/3}$ (Kirsch and Fuchs, 1968).

That there is such satisfactory agreement between the physically realistic model used in this paper and the more hypothetical cell models of Happel and Kuwabara does seem to indicate that these latter models possess considerable fundamental significance (the Happel model in particular) despite the fact that they do not actually represent any real physical situation. However, the overwhelming advantage of these cell models is that they are very easy to apply and provide good, physically consistent predictions in numerous diversified applications [for example, see Levine and Neale (1974) for an important application involving electrokinetic flow].

NOTATION

k = permeability of fiber mat
p = fluid pressure
$[r, \theta, z]$ = cylindrical coordinates
R = radius of reference fiber
S = outer radius of void shell
u_r, u_θ = velocity components
U^* = superficial flow velocity through mat
x = principal flow direction
W = drag force per unit length of fiber

Greek Letters

α = fiber mat concentration = R^2/S^2
μ = fluid viscosity
χ = dimensionless radial coordinate = r/R
ψ = stream function
τ = shear stress
ξ = dimensionless permeability = k/R^2
$*$ = denotes averaged quantities within porous medium

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Separation of Boron Isotopes by Direct Mode Thermal Parametric Pumping

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Results of boron isotope separation by direct mode thermal parametric pumping are presented in this R & D Note. Separation is based on the exchange reaction between gaseous BF_3 and solid $BF_3 \cdot DMSO$ (dimethyl sulfoxide) which can be written as

$$B^{10}F_3 + B^{11}F_3 \cdot DMSO \leftrightarrows B^{11}F_3 + B^{10}F_3 \cdot DMSO \quad (1)$$

with an equilibrium constant

$$K = \frac{[B^{11}F_3][B^{10}F_3 \cdot DMSO]}{[B^{10}F_3][B^{11}F_3 \cdot DMSO]} \quad (2)$$

This value is the ratio of B^{10}/B^{11} in the solid to that in the fluid phase. Parametric pumping separation is predicted on the variation of K with temperature so initial effort was focused on measuring this variation.

Palko et al. (1958, 1961, 1967), interested in obtaining the $_5B^{10}$ isotope because of its large neutron capture

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